2020 B

march 25

Green's theorem. Let D be a region enclosed by a simple, closed, piecewise smooth curve C. Let $\vec{F}$ be a smooth v.f. in D. Then

$$\oint_C M\,dx + N\,dy = \iint_D (N_x - M_y)\, dA(x,y), \text{ where}$$

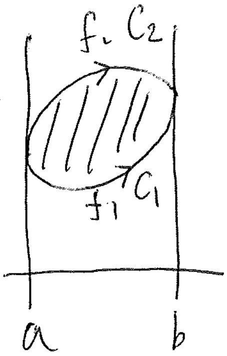
$\vec{F} = M\hat{i} + N\hat{j}$ and C is in anticlockwise direction.

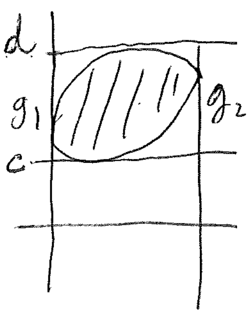
We verify the theorem for C of a special form.

Namely,

$$D = \{(x,y) : f_1(x) \le y \le f_2(x),\ a \le x \le b,\ f_1(a) = f_2(a),\ f_1(b) = f_2(b)\}$$

$$= \{(x,y) : g_1(y) \le x \le g_2(y),\ c \le y \le d,\ g_1(c) = g_2(c),\ g_1(d) = g_2(d)\}$$





claim:

$$\iint_D \frac{\partial M}{\partial y}\, dA = -\oint_C M\,dx$$

$$\iint_D \frac{\partial N}{\partial x}\, dA = \oint_C N\,dy$$

By adding up, we get Green's thm.

Now,
$$\oint_C M\,dx = \int_{C_1} M\,dx + \int_{-C_2} M\,dx$$

$$= \int_{C_1} M\,dx - \int_{C_2} M\,dx$$

where $C_1 : x \mapsto (x, f_1(x))$, $x \in [a,b]$

$C_2 : x \mapsto (x, f_2(x))$, $x \in [a,b]$.

$C = C_1 - C_2$.

$$\int_{C_1} M\,dx = \int_a^b M(x, f_1(x))\,dx ,$$

$$\int_{C_2} M\,dx = \int_a^b M(x, f_2(x))\,dx$$

$$\therefore \oint_C M\,dx = \int_a^b (M(x, f_1(x)) - M(x, f_2(x))\,dx .$$

On the other hand,

$$\iint_D M_y = \int_a^b \int_{f_1(x)}^{f_2(x)} \frac{\partial}{\partial y} M\,dy\,dx = \int_a^b (M(x, f_2(x)) - M(x, f_1(x)))\,dx$$

$$\therefore \iint_D \frac{\partial M}{\partial y}\,dA = -\oint_C M\,dx .$$

Similarly can check the other identity. #

We'll discuss 4 Consequences of Green's theorem.

The 1st one was discussed last time, ie, to use double integral to replace line integrals.

One more example will be given.

First, a reformulation:

$M \to -N$

$N \to M$

get

$$\iint_D (M_x + N_y)\,dA = \oint_C -N\,dx + M\,dy .$$

(normal form of Green's thm)

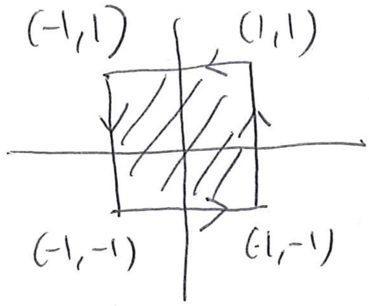
$=$ the flux of $\vec{F}$ across C

old one

$$\iint_D (N_x - M_y)\, dA = \oint_C M\,dx + N\,dy$$

$$= \text{the circulation of } \vec{F} \text{ around } C.$$

e.g. Find the flux of $2e^{xy}\hat{i} + y^3\hat{j}$ around the square



$$\text{flux} = \oint_C -N\,dx + M\,dy$$

$$= \iint_D (M_x + N_y)\,dA$$

$$= \iint_D (2y e^{xy} + 3y^2)\,dA$$

$$= \int_{-1}^{1}\int_{-1}^{1} (2y e^{xy} + 3y^2)\,dx\,dy$$

$$= \int_{-1}^{1} (2e^{y} - 2e^{-y} + 6y^2)\,dy = 4 \ \#$$

A

$\boxed{\text{Consequence 2}}$ (theoretical one)

We know

$$\vec{F} \text{ conservative} \Longrightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \text{(component test)}$$

but

$$\vec{F} \text{ conservative} \;\not\Longleftarrow\; \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

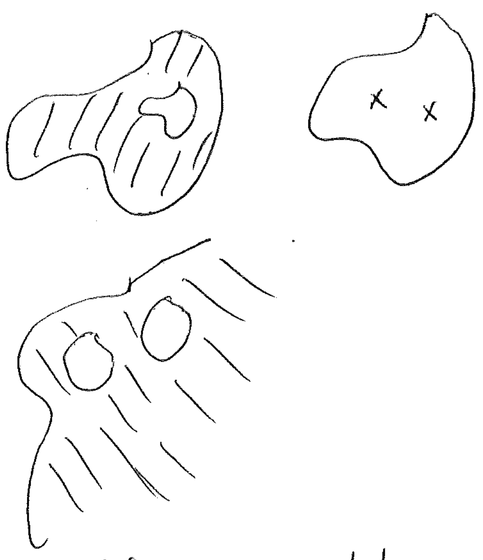
e.g. $\vec{F} = \dfrac{-y}{x^2+y^2}\hat{i} + \dfrac{x}{x^2+y^2}\hat{j}$.

Therem $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Longleftrightarrow \vec{F}$ conservative

when $\vec{F}$ is a smooth v.f. in a simply-connected region D

$D \subset \mathbb{R}^n$ is called simply-connected if every closed curve in D can be deformed continuously into a point when the whole process happens inside D.

When $n = 2$, a simply-connected region is the one without holes, nor punctured.



all 3 No good!

Pf. It suffices to show

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow \oint_C \vec{F} \cdot d\vec{r} = 0$$

$\forall$ closed C.

Let C be a simple closed curve in D. As D has no holes / punctured, the region enclosed by C, R, is contained in D. Hence $\vec{F}$ is well-def. in R.

Green's thm :

$$\oint_C M\,dx + N\,dy = \iint_R (N_x - M_y)\,dA$$

$$= 0.$$

So the loop property holds for all closed, simple curves C. It is not hard to see that it also holds for all closed curves. As loop property is equivalent to $\vec{F}$ being conservative, done #

Consequence (3). An area formula.

Take $N(x,y) = x$, $M(x,y) = 0$ ∴ Green's thm,

$$\oint_C 0\,dx + x\,dy = \iint_D (1-0)\,dA$$

$$\therefore\ |D| = \oint x\,dy.$$

Take $N(x,y) = 0$, $M(x,y) = -y$,

$$|D| = -\oint_C y\,dx$$

$$\therefore\ |D| = \frac{1}{2}\oint_C x\,dy - y\,dx \qquad \text{(area formula)}$$

in a more symmetric form.

e.g. Find the area of $x^2+y^2 \le R^2$.

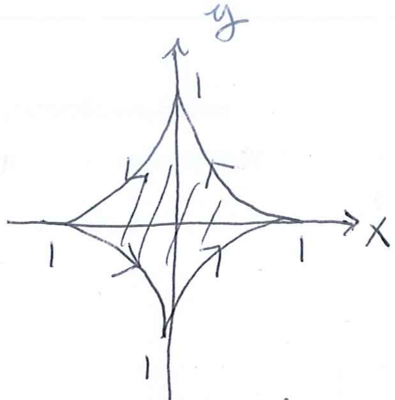
Choose $\vec{r}(t) = R\cos t\,\hat{i} + R\sin t\,\hat{j}$, $t \in [0, 2\pi]$.

$$\begin{aligned}|D| &= \text{area of the disk} \\ &= \frac{1}{2}\int_0^{2\pi} R\cos t\,(R\cos t) - R\sin t\,(-R\sin t)\,dt \\ &= \frac{1}{2}\int_0^{2\pi} R^2(\cos^2 t + \sin^2 t)\,dt \\ &= \pi R^2.\end{aligned}$$

e.g. Find the area of the region enclosed by the astroid $x(t) = \cos^3 t$, $y(t) = \sin^3 t$, $t \in [0, 2\pi]$

$x'(t) = -3\cos^2 t \sin t$

$y'(t) = 3\sin^2 t \cos t$



$$|D| = \frac{1}{2}\int_0^{2\pi} -y\,dx + x\,dy$$

$$= \frac{1}{2}\int_0^{2\pi} -\sin^3 t(-3\cos^2 t \sin t) + \cos^3 t(3\sin^2 t \cos t)\ dt$$

$$= \frac{3}{2}\int_0^{2\pi} \cos^2 t \sin^4 t + \sin^2 t \cos^4 t\ dt$$

$$= \frac{3}{2}\int_0^{2\pi} \cos^2 t \sin^2 t\,dt$$

$$= \frac{3}{8}\int_0^{2\pi} (\sin 2t)^2\,dt = \frac{3}{8}\frac{1}{2}\int_0^{2\pi} (1-\cos 4t)\,dt$$

$$= \frac{3}{16}\times 2\pi = \frac{3\pi}{8}\ \#$$

<u>Note.</u> in (x,y)-coordinates, the curve is given by $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1$, so over $x \in [0,1]$, $y(x) = (1-x^{2/3})^{3/2}$.

$$\text{area} = |D| = 4\int_0^1 (1-x^{2/3})^{3/2}\,dx,$$ but this is hard.

$\boxed{\text{Consequence (4)}}$. Localize flux and circulation.

Let $(x,y) \in D$. For curve C enclosing (x,y),

the circulation around C

$$= \oint M\,dx + N\,dy$$

$$= \iint_D (N_x - M_y)\,dA$$

As C shrinks down to (x,y), $|D| \to 0$, we see

$$\lim_{|D|\to 0} \frac{1}{|D|}\oint_C M\,dx + N\,dy = \lim_{|D|\to 0} \iint_D (N_x - M_y)\,dA$$

$$= (N_x - M_y)(x,y)$$

This motivate to define, the circulation (or the curl) of $\vec{F}$ at (x,y) to be,

$$(N_x - M_y)(x,y).$$

Similarly, define the flux (or the divergence) of $\vec{F}$ at (x,y) to be

$$(M_x + N_y)(x,y),$$

which is

$$\lim_{|D|\to 0} \frac{1}{|D|} \oint -N\,dx + M\,dy$$

$$= \lim_{|D|\to 0} \frac{1}{|D|} \iint_D (M_x + N_y)\,dA.$$

e.g. Find the divergence and curl of $\vec{F}$ at $(1,-2)$

$$\vec{F} = xy\,\hat{i} + \frac{x}{1+y}\,\hat{j}.$$

$$\text{div of } \vec{F} = M_x + N_y$$

$$= y - \frac{x}{(1+y)^2}$$

div of $\vec{F}$ at $(1,-2)$ is $-2 - \frac{1}{(1-2)^2} = -3$.

$$\text{curl of } \vec{F} = N_x - M_y = \frac{1}{1+y} - x$$

curl of $\vec{F}$ at $(1,-2)$ $= \frac{1}{1-2} - 1 = -2$ #